

# Uniqueness of Landau-Lifshitz Energy Frame in Relativistic Dissipative Hydrodynamics

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We show that the relativistic dissipative hydrodynamic equation derived from the relativistic Boltzmann equation by the renormalization-group method uniquely leads to the one in the energy frame proposed by Landau and Lifshitz, provided that the macroscopic-frame vector, which defines the local rest frame of the fluid velocity, is independent of the momenta of constituent particles, as it should. We argue that the relativistic hydrodynamic equations for viscous fluids must be defined on the energy frame if it is consistent with the underlying relativistic kinetic equation.

## I. INTRODUCTION

Theory of relativistic hydrodynamics for viscous fluids is a powerful mean for analyzing the long-wavelength and low-frequency dynamics of many body systems, such as the hot and/or dense quark-gluon or hadronic matter created by relativistic heavy-ion collisions [1–4] and also various high-energy astrophysical phenomena [5, 6].

However, the relativistic dissipative hydrodynamic equation is still under debate in the fundamental level. Indeed, the following three problems may be noted: (A) There are ambiguities in the definition of the fluid velocity [7, 8], (B) In the Eckart (particle) frame, there arises an unphysical instabilities of the equilibrium state [9], and (C) the so-called first-order equations lack in causality [10–12].

As is easily thought of, it is a legitimate and natural way for obtaining the proper relativistic hydrodynamic equation, to start with the relativistic Boltzmann equation (RBE) which is Lorentz invariant and does not have stability nor causality problems [10]. The problem is how to obtain an asymptotic dynamics in the far-infrared long-wavelength limit of the RBE, and several reduction methods [13–15] have been developed with some but not a complete success.

Recently, for examining the first two problems, (A) and (B), Ohnishi and the present authors [16, 17] applied the renormalization-group (RG) method [18–21] as a powerful reduction theory of dynamics to the RBE and succeeded in deriving a generic form of relativistic dissipative hydrodynamic equations with the stable equilibrium state. A key ingredient in the derivation was introduction of a time-like Lorentz-covariant vector  $\mathbf{a}^\mu$ , with  $\mathbf{a}^0 > 0$ :  $\mathbf{a}^\mu$  specifies the macroscopic and covariant coordinate system where the local rest frame of the fluid velocity  $u^\mu$  is defined, and is called the macroscopic-frame vector.  $\mathbf{a}^\mu$  could depend on the momenta  $p^\mu$  of constituent particles of the system as well as the space-time coordinate  $x^\mu$ . In fact, Ohnishi and the present authors [16, 17] could manage to derive the hydrodynamic equation in the Eckart (particle) frame only by making  $\mathbf{a}^\mu$  have a  $p^\mu$

dependence. In retrospect, the possible momentum dependence of  $\mathbf{a}^\mu$ , however, may not be legitimate for  $\mathbf{a}^\mu$  to play a macroscopic-frame vector, because it means that the macroscopic space-time is defined for respective particle states with a definite energy-momentum, and may lead to a difficulty in the physical interpretation of the space and time in which the hydrodynamics is defined. Thus, we are lead to require that  $\mathbf{a}^\mu$  should be independent of  $p^\mu$  and time-like vector with the Lorentz covariance.

In this paper, taking this requirement from the outset, we shall examine the outcomes and reach a significant conclusion that the relativistic hydrodynamics consistent with the underlying kinetic equation should be uniquely the one in the Landau-Lifshitz (energy) frame, but not in other frames. This means also that the so-called matching conditions for selecting the energy frame but not other frames [10] is uniquely derived from the underlying kinetic equation [22].

This paper is organized as follows: After a brief account of the basic properties of the relativistic Boltzmann equation (RBE), we show that  $\mathbf{a}^\mu$  that is independent of the momentum  $p^\mu$  must be naturally proportional to the fluid velocity  $u^\mu$ ;  $\mathbf{a}^\mu = b u^\mu$ . Furthermore, we show that the “normalization” factor  $b$  can be made unity without loss of generality. Then, we apply the RG method to derive the relativistic hydrodynamics from the RBE with  $\mathbf{a}^\mu = u^\mu$ , and show that the resulting hydrodynamic equation is uniquely in the energy frame. We also clarify the meaning of the matching condition in terms of the inner product for the distribution functions. The last section is devoted to a summary and concluding remarks on the cases of multi-component systems and the extended thermodynamics.

## II. RELATIVISTIC BOLTZMANN EQUATION

To make the presentation self-contained, we first summarize the basic facts about the relativistic Boltzmann equation (RBE) [10] very briefly. The RBE is an evo-

lution equation of the one-particle distribution function  $f_p(x)$  defined in the phase space  $(x, p)$ ,

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x), \quad (\text{II.1})$$

with  $p^\mu$  being the four-momentum of the on-shell particle, *i.e.*,  $p^\mu p_\mu = p^2 = m^2$  and  $p^0 > 0$ . The  $C[f]_p(x)$  in the right hand side denotes the collision operator,

$$C[f]_p(x) \equiv \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) \\ \times \left( f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x) \right), \quad (\text{II.2})$$

with  $\omega(p, p_1 | p_2, p_3)$  being the transition probability, which has the symmetry property  $\omega(p, p_1 | p_2, p_3) = \omega(p_2, p_3 | p, p_1) = \omega(p_1, p | p_3, p_2) = \omega(p_3, p_2 | p_1, p)$  and respects the energy-momentum conservation  $\omega(p, p_1 | p_2, p_3) \propto \delta^4(p + p_1 - p_2 - p_3)$ .

Thanks to the above symmetry property and the energy-momentum conservation, the function  $\varphi_p(x) = a(x) + p^\mu b_\mu(x)$  is found to be a collision invariant;

$$\sum_p \frac{1}{p^0} \varphi_p(x) C[f]_p(x) = 0, \quad (\text{II.3})$$

where  $a(x)$  and  $b_\mu(x)$  being arbitrary functions of  $x$ . On account of Eq. (II.3), we have formally the balance equations for the energy-momentum tensor  $T^{\mu\nu}(x) = \sum_p \frac{1}{p^0} p^\mu p^\nu f_p(x)$  and the particle current  $N^\mu(x) = \sum_p \frac{1}{p^0} p^\mu f_p(x)$  as follows;  $\partial_\nu T^{\mu\nu}(x) = 0$  and  $\partial_\mu N^\mu(x) = 0$ , respectively. Despite the appearance, these equations do not have any dynamical information before the evolution of  $f_p(x)$  has been obtained from Eq. (II.1).

The entropy current is defined by  $S^\mu(x) \equiv -\sum_p \frac{1}{p^0} p^\mu f_p(x) (\ln(2\pi)^3 f_p(x) - 1)$ , which is generically non-conserving;

$$\partial_\mu S^\mu(x) = -\sum_p \frac{1}{p^0} C[f]_p(x) \ln(2\pi)^3 f_p(x), \quad (\text{II.4})$$

due to Eq.(II.1). One sees that  $S^\mu$  becomes a conserved quantity only when  $\ln(2\pi)^3 f_p(x)$  is a collision invariant,  $\ln(2\pi)^3 f_p(x) = a(x) + p^\mu b_\mu(x)$ . Accordingly, an entropy-conserving distribution function can be cast into the form of the local equilibrium distribution function (the Jüttner function [23]) in terms of the local temperature  $T(x)$ , chemical potential  $\mu(x)$ , and time-like fluid velocity  $u^\mu(x)$  as  $f_p(x) = (2\pi)^{-3} \exp[(\mu(x) - p^\mu u_\mu(x))/T(x)] \equiv f_p^{\text{eq}}(x)$ , with  $u^\mu(x) u_\mu(x) = 1$  and  $u^0(x) > 0$ . We also note that the collision operator identically vanishes for the *local* equilibrium distribution function  $f_p^{\text{eq}}(x)$ ;

$$C[f^{\text{eq}}]_p(x) = 0, \quad (\text{II.5})$$

due to the energy-momentum conservation in the collision process.

### III. MACROSCOPIC-FRAME VECTOR

To implement the task to solve the RBE in the hydrodynamic regime, it is customary [10, 16, 17] to introduce a time-like Lorentz vector  $\mathbf{a}^\mu$ , with  $\mathbf{a}^0 > 0$ . We call  $\mathbf{a}^\mu$  the macroscopic-frame vector, which defines the covariant and macroscopic coordinate system  $(\tau, \sigma^\mu)$  from the space-time coordinate  $x^\mu$  as

$$d\tau = \mathbf{a}^\mu dx_\mu, \quad (\text{III.1})$$

$$d\sigma^\mu = \left( g^{\mu\nu} - \frac{\mathbf{a}^\mu \mathbf{a}^\nu}{\mathbf{a}^2} \right) dx_\nu. \quad (\text{III.2})$$

It is noted that  $(\tau, \sigma^\mu)$  is the so-called local Lorentz frame when  $\mathbf{a}^\mu$  is dependent on  $x^\mu$ .

We now show that  $\mathbf{a}^\mu$  must be proportional to the fluid velocity  $u^\mu$ , provided that  $\mathbf{a}^\mu$  should be independent of the momentum  $p^\mu$  and time-like vector with the Lorentz covariance. Noting that  $u^\mu$  and  $\partial^\mu$  are the only available Lorentz vectors at hand, we see that the generic form of a Lorentz-covariant vector reads

$$\mathbf{a}^\mu = A_1 u^\mu + A_2 \partial^\mu T + A_3 \partial^\mu \mu + A_4 u^\nu \partial_\nu u^\mu, \quad (\text{III.3})$$

where  $A_i$  with  $i = 1, 2, 3, 4$  is an arbitrary Lorentz-scalar function of the temperature and the chemical potential;  $A_i = A_i(T, \mu)$ . Now the derivative  $\partial^\mu$  can be decomposed into the time-like and space-like components as  $\partial^\mu = u^\mu u^\nu \partial_\nu + (g^{\mu\nu} - u^\mu u^\nu) \partial_\nu \equiv u^\mu D + \nabla^\mu$ , where  $D \equiv u^\nu \partial_\nu$  and  $\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$  with  $\Delta^{\mu\nu} = (g^{\mu\nu} - u^\mu u^\nu)$  being the projection operator onto a space-like vector orthogonal to the time-like vector  $u^\mu$ . Then Eq.(III.3) is also decomposed as

$$\mathbf{a}^\mu = (A_1 + A_2 DT + A_3 D\mu) u^\mu \\ + A_2 \nabla^\mu T + A_3 \nabla^\mu \mu + A_4 Du^\mu. \quad (\text{III.4})$$

But since  $\mathbf{a}^\mu$  should be time-like, we have

$$\mathbf{a}^\mu = (A_1 + A_2 DT + A_3 D\mu) u^\mu \equiv b(T, \mu) u^\mu, \quad (\text{III.5})$$

with  $b(T, \mu) > 0$ . Here, we have used the fact that  $\nabla^\mu T$ ,  $\nabla^\mu \mu$ , and  $Du^\mu$  are space-like because  $\Delta^{\mu\nu} \nabla_\nu = \nabla^\mu$  and  $\Delta^{\mu\nu} Du_\nu = Du^\mu$ .

With the use of this macroscopic-frame vector, the covariant and macroscopic coordinate system  $(\tau, \sigma^\mu)$  reads  $d\tau = b(T, \mu) u^\mu dx_\mu$  and  $d\sigma^\mu = \Delta^{\mu\nu} dx_\nu$ . In fact, the “normalization” factor  $b(T, \mu)$  is a redundant degree of freedom for the dynamics because the  $b(T, \mu)$  can be always made unity by converting  $\tau$  into the new temporal coordinate  $\tau'$  as  $d\tau' \equiv b(T, \mu)^{-1} d\tau = u^\mu dx_\mu$ . From now on, we thus set

$$\mathbf{a}^\mu = u^\mu. \quad (\text{III.6})$$

In terms of the new coordinates  $(\tau, \sigma^\mu)$ , Eq.(II.1) is rewritten as

$$\frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot u} C[f]_p(\tau, \sigma) \\ - \varepsilon \frac{1}{p \cdot u} p \cdot \nabla f_p(\tau, \sigma), \quad (\text{III.7})$$

where  $\partial/\partial\tau \equiv D$  and  $\partial/\partial\sigma_\mu \equiv \nabla^\mu$ . Here the small parameter  $\varepsilon$ , which will be set back to unity eventually, represents the non-uniformity of space. This seemingly mere rewrite of the equation reflects a physical assumption that only the spatial inhomogeneity is the origin of the dissipation. We note that the RG method applied to the non-relativistic case with this assumption successfully leads to the Navier-Stokes equation [25], which means that the present approach [16, 17] is a covariantization of the non-relativistic case.

#### IV. REDUCTION TO HYDRODYNAMIC EQUATION

In this section, we derive the relativistic hydrodynamics through solving the converted Boltzmann equation (III.7) in the RG method.

We should note that some of formulas presented in this section may be found in the previous papers [16, 17] in a different context and hence in a different order. Thus, the presentation will be a brief reorganization of them; we refer to Ref. [17] for the detailed reasoning.

##### A. Hydrodynamics from relativistic Boltzmann equation by RG method

In the RG method developed in Ref.'s [16, 17, 19, 20], we first try to obtain the perturbative solution  $\tilde{f}_p$  to Eq. (III.7) around the arbitrary initial time  $\tau = \tau_0$  with the initial value  $f_p(\tau_0, \sigma)$ ;  $\tilde{f}_p(\tau = \tau_0, \sigma; \tau_0) = f_p(\tau_0, \sigma)$ . Note that the solution depends on the initial time  $\tau_0$  at which  $\tilde{f}_p(\tau = \tau_0, \sigma; \tau_0)$  is supposed to be on an exact solution. We expand the initial value as well as the solution with respect to  $\varepsilon$  as follows:

$$\tilde{f}_p(\tau, \sigma; \tau_0) = \sum_{i=0}^{\infty} \varepsilon^i \tilde{f}_p^{(i)}(\tau, \sigma; \tau_0), \quad (\text{IV.1})$$

$$f_p(\tau_0, \sigma) = \sum_{i=0}^{\infty} \varepsilon^i f_p^{(i)}(\tau_0, \sigma). \quad (\text{IV.2})$$

The zeroth-order equation is

$$\frac{\partial}{\partial\tau} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = \frac{1}{p \cdot u} C[\tilde{f}^{(0)}]_p(\tau, \sigma; \tau_0). \quad (\text{IV.3})$$

Since we are interested in the slow motion realized asymptotically for  $\tau \rightarrow \infty$ , we take the stationary solution satisfying  $\partial\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)/\partial\tau = 0$ , which demands that

$$C[\tilde{f}^{(0)}]_p(\tau, \sigma; \tau_0) = 0, \quad \forall \sigma. \quad (\text{IV.4})$$

Such a solution is given by a local equilibrium distribution function, *i.e.*, the Jüttner distribution function,

$$\begin{aligned} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) &= \frac{1}{(2\pi)^3} \exp \left[ \frac{\mu(\sigma; \tau_0) - p^\mu u_\mu(\sigma; \tau_0)}{T(\sigma; \tau_0)} \right] \\ &\equiv f_p^{\text{eq}}(\sigma; \tau_0), \end{aligned} \quad (\text{IV.5})$$

with  $u^\mu(\sigma; \tau_0) u_\mu(\sigma; \tau_0) = 1$ . We note that the integration constants  $T(\sigma; \tau_0)$ ,  $\mu(\sigma; \tau_0)$ , and  $u_\mu(\sigma; \tau_0)$  are independent of  $\tau$  but may depend on  $\tau_0$  as well as  $\sigma$ .

The first-order equation is given by

$$\frac{\partial}{\partial\tau} \tilde{f}_p^{(1)}(\tau) = \sum_q A_{pq} \tilde{f}_q^{(1)}(\tau) + F_p, \quad (\text{IV.6})$$

where

$$F_p \equiv -\frac{1}{p \cdot u} p \cdot \nabla f_p^{\text{eq}}. \quad (\text{IV.7})$$

with  $A_{pq}$  being a matrix element of the linearized collision operator  $A$ ,

$$(A)_{pq} = A_{pq} \equiv \frac{1}{p \cdot u} \frac{\partial}{\partial f_q} C[f]_p \bigg|_{f=f^{\text{eq}}}. \quad (\text{IV.8})$$

We now show that the linearized collision operator  $A$  has remarkable spectral properties [16, 17], which are essential for the following analysis. We first convert  $A$  to another linear operator,

$$L \equiv (f^{\text{eq}})^{-1} A f^{\text{eq}}, \quad (\text{IV.9})$$

with  $(f^{\text{eq}})_{pq} \equiv f_p^{\text{eq}} \delta_{pq}$ , which is diagonal. Next, an inner product is introduced for arbitrary nonzero vectors  $\varphi$  and  $\psi$  as

$$\langle \varphi, \psi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} \varphi_p \psi_p, \quad (\text{IV.10})$$

which satisfies the positive-definiteness of the norm as  $\langle \varphi, \varphi \rangle = \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} (\varphi_p)^2 > 0$ , for  $\varphi_p \neq 0$ , due to  $(p \cdot u) > 0$  because of both  $p^\mu$  and  $u^\mu$  being time-like vectors.

Then, as is shown in Ref.'s [16, 17], the linearized collision operator is semi-negative definite and has five zero modes given by

$$\varphi_{0p}^\alpha \equiv \begin{cases} p^\mu & \text{for } \alpha = \mu = 0 \sim 3, \\ 1 \times m & \text{for } \alpha = 4. \end{cases} \quad (\text{IV.11})$$

The functional subspace spanned by the five zero modes is called the  $P_0$  space and the projection operator to it is denoted by  $P_0$ ;  $[P_0 \psi]_p \equiv \varphi_{0p}^\alpha \eta_{0\alpha\beta}^{-1} \langle \varphi_0^\beta, \psi \rangle$ , where  $\eta_{0\alpha\beta}^{-1}$  is the inverse matrix of the the  $P_0$ -space metric matrix  $\eta_0^{\alpha\beta}$  defined by  $\eta_0^{\alpha\beta} \equiv \langle \varphi_0^\alpha, \varphi_0^\beta \rangle$ . We also call the complement to the  $P_0$  space the  $Q_0$  space and introduce  $Q_0 \equiv 1 - P_0$ . In the following, we also use the modified projection operators defined by  $\bar{P}_0 = f^{\text{eq}} P_0 (f^{\text{eq}})^{-1}$  and  $\bar{Q}_0 = f^{\text{eq}} Q_0 (f^{\text{eq}})^{-1}$ .

Then, the perturbative solution up to the second order reads  $\tilde{f}_p(\tau, \sigma; \tau_0) = \tilde{f}_p^{(0)} + \varepsilon \tilde{f}_p^{(1)} + \varepsilon^2 \tilde{f}_p^{(2)} + O(\varepsilon^3)$ , where

$$\tilde{f}_p^{(1)} = (\tau - \tau_0) \bar{P}_0 F - A^{-1} \bar{Q}_0 F, \quad (\text{IV.12})$$

and we refer to Ref. [17] for a lengthy formula of  $\tilde{f}_p^{(2)}$ , for the sake of space. We note that secular terms are present in  $\tilde{f}_p^{(i)}$  ( $i = 1, 2, \dots$ ), which is caused by the zero modes of the linearized collision operator  $A$ .

Remarks are in order here: In the usual approaches [10, 17] to derive hydrodynamic equations from RBE, so-called matching conditions corresponding to a desired frame are imposed on the higher-order terms from the outset, and the hydrodynamic equation in the desired frame including Eckart one is formally obtained. It is noteworthy that the matching conditions are also a condition to forbid the appearance of secular terms [10]; we refer to Ref. [17] for some problematic aspects of the matching conditions. In the present approach, on the other hand, we have so far just solved the Boltzmann equation in the perturbation theory in a straightforward way without imposing any matching conditions, and the resulting higher-order terms contain secular terms, which apparently invalidates the perturbative expansion for  $\tau$  away from the initial time  $\tau_0$ .

The key idea in the RG method, however, lies in the fact that we can utilize these apparently problematic secular terms to obtain an asymptotic solution valid in a global domain [19, 20]. Indeed, one can make the following geometrical interpretation of the perturbative solution constructed around arbitrary initial time  $\tau_0$ : That is, we have constructed a family of curves  $\tilde{f}_p(\tau, \sigma; \tau_0)$  parameterized with  $\tau_0$ , which curves are all supposed to be on the exact solution  $f_p(\sigma; \tau)$  at  $\tau = \tau_0$  up to  $O(\varepsilon^3)$ , although they are admittedly only valid for  $\tau$  near  $\tau_0$  locally. Then, the *envelope curve* of the family of curves will give a global solution in our asymptotic situation, which is shown indeed to be the case [19, 20]. According to the classical theory of envelopes, the envelope that is in contact with any curve in the family at  $\tau = \tau_0$  is obtained [19] by

$$\left. \frac{d}{d\tau_0} \tilde{f}_p(\tau, \sigma; \tau_0) \right|_{\tau=\tau_0} = 0. \quad (\text{IV.13})$$

The derivative with respect to  $\tau_0$  hits on the hydrodynamic variables, and hence we have the evolution equation of them, which is identified with the hydrodynamic equation [16, 25]. We also note that the invariant manifold [24] corresponding to the hydrodynamics in the functional space of the distribution function is explicitly obtained as an envelope function [16, 17]:  $f_{Ep}(\tau, \sigma) = \tilde{f}_p(\tau, \sigma; \tau_0 = \tau)$ , the explicit form of which is referred to Ref.'s [16, 17]. We note that this solution is valid in a global domain of time in the asymptotic region [17].

Putting back  $\varepsilon$  to 1, Eq.(IV.13) is reduced to the following form in this approximation,

$$\sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[ (p \cdot u) \frac{\partial}{\partial \tau} + p \cdot \nabla \right] (f_p^{\text{eq}} + \delta f_p^{(1)}) = 0. \quad (\text{IV.14})$$

where  $\delta f_p^{(1)}$  denotes the first-order correction to the distribution function

$$\delta f_p^{(1)} \equiv -[A^{-1} \bar{Q}_0 F]_p. \quad (\text{IV.15})$$

If one uses the identity  $(p \cdot u) \partial / \partial \tau + p \cdot \nabla = p^\mu \partial_\mu$ , Eq.(IV.14) is found to have the following form

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0, \quad (\text{IV.16})$$

with  $T^{\mu\nu} = T^{(0)\mu\nu} + \delta T^{\mu\nu}$  and  $N^\mu = N^{(0)\mu} + \delta N^\mu$ . Here, the zero-th order terms read  $T^{(0)\mu\nu} \equiv \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{eq}} = e u^\mu u^\nu - p \Delta^{\mu\nu}$  and  $N^{(0)\mu} \equiv \sum_p \frac{1}{p^0} p^\mu f_p^{\text{eq}} = n u^\mu$ , with  $e$ ,  $p$ , and  $n$  being the internal energy, pressure, and particle-number density for the relativistic ideal gas, respectively, while the dissipative parts are given by

$$\delta T^{\mu\nu} \equiv \sum_p \frac{1}{p^0} p^\mu p^\nu \delta f_p^{(1)}, \quad (\text{IV.17})$$

$$\delta N^\mu \equiv \sum_p \frac{1}{p^0} p^\mu \delta f_p^{(1)}. \quad (\text{IV.18})$$

Note that the dissipative terms are due to the deviation  $\delta f_p^{(1)}$  of the distribution function from the local one. It is well known that the local distribution function only gives the (relativistic) Euler equation without dissipation.

## B. Uniqueness of Landau-Lifshitz frame

In this subsection, we present the dissipative parts  $\delta T^{\mu\nu}$  and  $\delta N^\mu$ , explicitly, and discuss their properties. An explicit evaluation of Eq.'s (IV.17) and (IV.18) together with (IV.15) gives [16, 17]

$$\delta T^{\mu\nu} = \zeta \Delta^{\mu\nu} \nabla \cdot u + 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (\text{IV.19})$$

$$\delta N^\mu = \lambda \frac{1}{\hat{h}^2} \nabla^\mu \frac{\mu}{T}, \quad (\text{IV.20})$$

respectively, with  $\Delta^{\mu\nu\rho\sigma} \equiv 1/2 \cdot (\Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - 2/3 \cdot \Delta^{\mu\nu} \Delta^{\rho\sigma})$ . Here,  $\hat{h}$  denotes the reduced enthalpy per particle. The bulk and shear viscosities and the thermal conductivity are denoted by  $\zeta$ ,  $\eta$  and  $\lambda$ , respectively. One readily finds that these formulas completely agree with those proposed by Landau and Lifshitz [8]. Indeed, the respective dissipative parts  $\delta T^{\mu\nu}$  and  $\delta N^\mu$  in Eq.'s (IV.19) and (IV.20) meet Landau and Lifshitz's constraints

$$\delta e \equiv u_\mu \delta T^{\mu\nu} u_\nu = 0, \quad (\text{IV.21})$$

$$Q_\mu \equiv \Delta_{\mu\nu} \delta T^{\nu\rho} u_\rho = 0, \quad (\text{IV.22})$$

$$\delta n \equiv u_\mu \delta N^\mu = 0, \quad (\text{IV.23})$$

which are imposed in a heuristic way in the phenomenological derivation [8]. Thus, we have found that the frame on which the fluid velocity is defined necessarily becomes the Landau-Lifshitz (energy) frame, if the hydrodynamics is to be consistent with the underlying RBE.

Next, let us examine the underlying meaning of Eq.'s (IV.21) to (IV.23) in terms of the distribution function. As was mentioned above, these equations are usually just imposed [10] to the higher-order terms of the distribution function as the matching conditions without any foundation to select the hydrodynamic equation in the energy frame. We shall clarify that these conditions are equivalent to the orthogonality condition for the excited modes expressed in terms of the inner product [16, 17] and hence an inevitable *consequence* for the relativistic hydrodynamics in our analysis which is free from any ansatz.

We first note that Eq.'s (IV.17) and (IV.18) can be rewritten as

$$\delta T^{\mu\nu} = \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{eq}} \bar{\phi}_p, \quad (\text{IV.24})$$

$$\delta N^\mu = \sum_p \frac{1}{p^0} p^\mu f_p^{\text{eq}} \bar{\phi}_p, \quad (\text{IV.25})$$

with

$$\bar{\phi}_p = (f_p^{\text{eq}})^{-1} \delta f_p^{(1)} = -[L^{-1} Q_0 (f_p^{\text{eq}})^{-1} F]_p, \quad (\text{IV.26})$$

which belongs to the  $Q_0$  space and thus orthogonal to the zero modes,

$$\langle \varphi_0^\alpha, \bar{\phi} \rangle = 0 \text{ for } \alpha = 0, 1, 2, 3, 4. \quad (\text{IV.27})$$

Recalling the definition Eq. (IV.10) of the inner product, we see that Eq. (IV.27) with  $\alpha = \mu = 0, 1, 2, 3$  is reduced to

$$\begin{aligned} 0 &= \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} p^\mu \bar{\phi}_p = u_\nu \sum_p \frac{1}{p^0} p^\nu p^\mu f_p^{\text{eq}} \bar{\phi}_p \\ &= u_\nu \delta T^{\mu\nu}, \end{aligned} \quad (\text{IV.28})$$

which readily leads to Eq.'s (IV.21) and (IV.22). Quite similarly, Eq. (IV.27) with  $\alpha = 4$  is reduced to

$$\begin{aligned} 0 &= \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} \bar{\phi}_p = u_\nu \sum_p \frac{1}{p^0} p^\nu f_p^{\text{eq}} \bar{\phi}_p \\ &= u_\nu \delta N^\nu, \end{aligned} \quad (\text{IV.29})$$

which is nothing but Eq. (IV.23).

We emphasize again that the matching conditions for the energy frame are not imposed but uniquely obtained in our derivation from the underlying kinetic equation, without any assumptions nor ansatz. This facts may mean that the relativistic hydrodynamic equation for a viscous fluid must be defined in the energy frame, at least if it is consistent with the underlying kinetic equation.

## V. SUMMARY AND DISCUSSIONS

In this paper, we have shown that the renormalization-group (RG) derivation of the relativistic dissipative hydrodynamic equation as the infrared dynamics of the underlying relativistic Boltzmann equation (RBE) uniquely

leads to the one in the energy frame proposed by Landau and Lifshitz, provided that the macroscopic-frame vector, which covariantly defines the local rest frame of the fluid velocity, is independent of the momenta of constituent particles of the system, as it should.

In relation to other methods of the derivation of hydrodynamic equations based on the RBE, we note that we have not assumed any matching conditions [10] but uniquely got them in the energy frame from the underlying kinetic equation, and hence given the foundation for the matching conditions. Since any matching conditions are not imposed, secular terms due to the zero modes of the linearized collision operator appear inevitably in higher-order terms in our approach, but they are resummed away by the RG/envelope equation to give an asymptotic solution valid in a global domain.

We thus argue that the relativistic hydrodynamic equation for viscous fluids must be defined in the energy frame, if it is consistent with the underlying kinetic equation. Although the RBE which we have adopted as the kinetic equation is admittedly suitable only for a dilute gas, it is expected that the derived hydrodynamic equation itself and hence the uniqueness of the energy frame can be valid even for dense systems; this is found plausible if one recalls the universal nature of (non-relativistic) Navier-Stokes equation beyond dilute systems, although it can be also derived [13, 25] from the (non-relativistic) Boltzmann equation.

Although the present work is confined to the case of the so-called first-order equation for a system composed of a single component, the uniqueness of the energy frame for the relativistic hydrodynamics may keep valid for the multi-component systems [10] and the case of the so-called second-order causal relativistic hydrodynamics, *i.e.*, the extended thermodynamics [26]. In fact, we can show that the energy frame is the most natural frame for multi-component systems [27] and that the RG derivation [28] of the mesoscopic dynamics [30] of the RBE naturally leads to the extended thermodynamics in the energy frame. We can thus assert more firmly that the relativistic hydrodynamic equations for viscous fluids must be defined in the energy frame, if it is consistent with the underlying relativistic kinetic equation at all.

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